Math 33A Worksheet Week 5

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Exercise 1. Determine whether the following sets of vectors are linearly independent or linearly dependent:

(a) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

Exercise 2. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix $\begin{bmatrix} 4 & 2 & -1 \\ 0 & 1 & 0 \\ -8 & 0 & 2 \end{bmatrix}$.
Find a basis for $\ker A$. Find a basis for $\operatorname{im} A$. What are the dimensions of $\ker A$ and $\operatorname{im} A$?

Exercise 3. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies each of the following conditions, or explain why such a linear transformation doesn't exist:

(a) $\ker A = \{\vec{0}\}$, $\operatorname{im} A = \{\vec{0}\}$.

(b) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A = 0$

(c) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}\}$ and $\det A \neq 0$

(d) $\ker A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$, $\operatorname{im} A = \operatorname{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$.

Exercise 4. Find the kernel of the following matrices:

(a) $\begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$

(c) A for $A : \mathbb{R}^n \to \mathbb{R}^n$ invertible.

(d) A with row reduced echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$.

Challenge/Conceptual Problems

Exercise 5.
Let A be a $m \times n$ matrix, $v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ such that $Av = w$. Now suppose that v' is another vector in $\mathbb{R}^n$ such that $Av' = 0$, i.e. v' is in $\ker A$. What is $A(v + v')$?
(For a concrete example, Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $w = Av = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. Now if we let $v' = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so $Av' = 0$, what is $A(v + v') = A \begin{bmatrix} -1 \\ 2 \end{bmatrix}$?)

Exercise 6.
Let A be any matrix, and B be the RREF for A.

(a) Is $\ker A = \ker B$?

(b) Is $\operatorname{im} A = \operatorname{im} B$?
